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Capability of the BBC2008 yield criterion in predicting the earing profile in cup deep drawing simulations



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ABSTRACT

The paper deals with constitutive modeling of highly anisotropic sheet metals and presents FEM based earing predictions in a round cup drawing simulation of highly anisotropic aluminum alloys where more than four ears occur. For that purpose the BBC2008 yield criterion, which is a plane-stress yield criterion formulated in the form of a finite series, is used. Thus defined criterion can be expanded to retain more or less terms, depending on the amount of given experimental data. To be used in sheet metal forming simulations the constitutive model, derived in accordance with the associated flow theory of plasticity, has been implemented in a general purpose finite element code ABAQUS/Explicit via VUMAT subroutine, considering alternatively different number of parameters in the BBC2008 yield criterion, where possible number of parameters are any multiple of number 8. For the integration of the constitutive model the explicit NICE (Next Increment Corrects Error) integration scheme has been used. The CPU time consumption for an explicit deep drawing simulation, which is based on the developed constitutive model, has been proven to be, due to effectiveness of the used integration scheme, fully comparable to the performance experienced when the simulation is performed with ABAQUS built-in constitutive models and implicit integration schemes. Two aluminum alloys, namely AA5042-H2 and AA2090-T3, have been considered for a validation of the constitutive model. The respective BBC2008 model parameters have been identified for both allovs with a developed numerical procedure, based on a minimization of the specified cost function. For both materials, the simulation predictions based on the BBC2008 model prove to be in very good agreement with the experimental results. Further, in order to show the flexibility of the BBC2008 model in modeling of highly anisotropic sheet metal response, we have introduced a highly anisotropic fictitious material which yields, according to the theory, twelve ears in cup drawing. As it is shown in the paper the BBC2008 model is able to predict twelve ears in cup drawing simulation with the formulation containing 16 parameters for anisotropy description only. The flexibility and accuracy of the constitutive model together with the robust identification and integration procedure guarantee the applicability of the BBC2008 yield criterion in industrial applications.

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1. Introduction

Numerical simulation has become an important analysis tool in the manufacturing processes design. By shortening both the time required in the D&R stage and the time needed for a corresponding production implementation it contributes significantly to reducing the overall production costs and failure rate. Becoming in this regard indispensable, this computational instrument has extended during the last decades its applicability to various industrial fields. The consistency of the decisions made on the numerical simulation basis is highly dependent on the attained degree of the physical objectivity and numerical accuracy of the simulation. Many researchers involved in this domain focused consequently their efforts on improving both the quality of the theoretical models implemented in the simulation programs and efficiency of there applied computational strategies. Extensive past research performed in the field of sheet metal forming processes has proven that the adopted constitutive models have a strong influence on the reliability of the obtained numerical simulation results. In this paper we are analyzing, referring to the earing and height of

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investigated deep-drawn cylindrical cups, the influence of the yield surface model on the quality of the computed predictions.

The earing phenomenon has been noticed in the early applications of the rolled metallic sheets for the deep-drawing of cylindrical parts (beginning of the 19th century). The number of ears is usually four, but there are also situations when six or eight ears may occur at the upper edge of the drawn cups. A particular evolution of the earing process is generally dependent on the anisotropy of the sheet metal and specific lubrication conditions. A systematic analysis of this phenomenon was performed by Baldwin et al. (1945). These researchers were the first who noticed the similarity between the planar distribution of the anisotropy coefficient and the earing profile. On the basis of the experimental results reported by Baldwin et al. (1945), Bourne and Hill (1950) evaluated the dependency between the earing amplitude and the values of the planar anisotropy coefficients. Later on, during the 1960's, Blade and Pearson (1962/63), Wright (1965/66), Buschman, (1966) and other researchers studied the influence of several other parameters on the earing profile (geometry of the tools, gap between the die and punch surfaces, thickness of the sheet metal, drawing ratio, blank-holding force, etc.). In the same period of time, the earing phenomenon was also thoroughly investigated by several Russian researchers. The monograph published by Sevelev and Iakovlev (1972) provides a detailed description of their results. The cited work presents in a systematic manner the influence of various material and process parameters on the earing amplitude. It also gives an analytical formula that predicts the height and thickness of the ears on the basis of the anisotropy coefficient of the sheet metal. The theoretical results are also compared with experimental data obtained for different materials. The research started by Sevelev and Iakovlev has been continued in the 1980's by Grecinikovim (1985) who studied the influence of the temperature on the earing process in the case of steel, aluminum and brass. The results obtained by the Russian scientists have been presented in the monographs published by Matveeva (1987) and Zharkov (1995). An analytical relationship for the calculation of the earing profile on the basis of the plastic anisotropy coefficients has been also deduced by Hosford and Caddell (1983). Chung et al. (1996) improved that formula and used it in the case of different materials. They noticed the existence of a direct relation between the planar distribution of the plastic anisotropy coefficient and the earing profile, as well as between the planar distribution of the yield stress and the earing amplitude. Barlat et al. (1991b) have also developed a simple analytical model for the determination of the earing height using the planar distribution of the yield stress. Yoon et al. (2006) have extended that model and obtained a formula for the calculation of the cup height taking into account the dimensional characteristics of the cylindrical part and flat blank, as well as the plastic anisotropy. Later on, Yoon et al. have improved the accuracy of their model by combining it with the one previously proposed by Barlat et al. (1991b) and Yoon et al. (2006). The new analytical model (Yoon et al., 2008) defines a relationship between the earing profile and amplitude and the planar distribution of both the plastic anisotropy coefficient and yield stress. Recently, Yoon et al. (2011b) have developed a more accurate formulation that can be extended to the case of the ironing process. Mulder and Nagy, (2009), as well as Mulder et al. (2011) have also improved Yoon's model (Yoon et al., 2008) by taking the non-uniformity of the strain distribution on the surface of the cup into account. Another analytical approach for predicting the earing profile in the case of cylindrical cups, based on the slip-line theory, has been proposed by Jimma (1970/71), Sowerby and Johnson (1974), and Chen and Sowerby (1996). Their analysis is based on the plane strain hypothesis connected with Hill's anisotropic plasticity model.

Some researches use the crystallographic models for the earing profile prediction. The early results obtained with such methods are significant only from the qualitative point of view. The first paper dealing with the crystallographic prediction of the earing profile was published by Tucker in 1961 (Tucker, 1961). The model developed by Tucker is restricted to a single aluminum crystal. Recently Li et al. (2008) presented ear profiles in deep cup drawing, obtained by rate dependent crystal plasticity model, and show the comparison with experiments.

The development of the numerical techniques during the last three decades allows the earing prediction to be done by a corresponding finite element simulation of the considered sheet metal forming process. The quantitative analyses related to the accuracy of the numerical results show that the constitutive models, in particular yield criteria, used in the simulation, have a significant influence on the predicted earing profile. As a consequence, many researchers have focused their interest on the investigation of possibilities to improve the quality of the considered numerical predictions by adopting more realistic material models. With the aim of analyzing the earing profile the anisotropic yield criterion proposed by Hill in 1948 (Hill, 1948) was extensively used in the cup drawing process simulations. During the last two decades, more sophisticated yield criteria have been developed (Barlat and Lian, 1989; Barlat et al., 1991a; Barlat et al., 1997a; Barlat et al., 1997b; Barlat et al., 2000; Karafillis and Boyce, 1993; Ferron et al., 1994; Banabic et al., 2000; Cazacu and Barlat, 2001; Banabic et al., 2005; Comsa and Banabic, 2007; Yoshida et al., 2013). Some of those material models have been implemented in finite element programs and used for a prediction of the earing profile. From the publications devoted to this subject, we could mention the paper published by Yang and Kim (1986) on the numerical prediction of the earing profile using Hill 1948 yield criterion, as well as the papers published by Barlat et al. (1990), Chung and Shah (1992) and Habraken and Dautzenberg who used Barlat 1991 model (Barlat et al., 1991a). In general, the earing prediction obtained when adopting Barlat 1991 yield criterion exhibits a better accuracy. Cvitanic et al. (2008) have also compared the earing predictions of two plasticity models, namely Hill (1948) and Karafillis and Boyce (1993). The superior quality of the results obtained when using the Karafillis-Boyce model has been emphasized and explained by the authors. By considering three different yield criteria: Hill (1948), Barlat and Lian (1989), Gotoh (1977) and Hu et al. (2001) have thoroughly analyzed the influence of the anisotropy coefficients on the earing profile and its evolution. Their analysis has been focused on materials having a strong planar anisotropy. A comparative study on the accuracy of the predictions referring to the planar distribution of the uniaxial yield stress and the anisotropy coefficient and its relationship with the prediction of the earing profile has been performed by Soare et al. (2007) using Hill 1948 (Hill, 1948), Barlat 1996 (Barlat et al., 1997b) and Cazacu and Barlat (Cazacu and Barlat, 2001) yield criteria. A better quality of the earing predictions has been also noticed by Cosovici and Banabic (2005) in the case of the BBC 2000 yield criterion (Banabic et al., 2003) as compared to other constitutive models (Hill 1948 (Hill, 1948), Hill 1990 (Hill, 1990) and Barlat 1989 (Barlat and Lian, 1989)). Soare et al. (2008) have analyzed the predictive capabilities of the polynomial yield criteria Poly 6 and Poly 8 and concluded that the accuracy of the calculated earing profile depends on the quality of the anisotropy description and not on the mathematical formulation of the constitutive model. Comsa and Banabic (2007) have shown that the yield criteria having the capability to describe the anisotropic behavior of sheet metals in more detail generally provide better predictions of the earing profile. A systematic analysis of the topics related to the finite element prediction of earing of drawn cups has been performed in a series of papers by Yoon et al. (1995, 1998, 1999, 2000, 2004, 2006, 2011b, 2011a). Yoon et al. have focused their research interest on

investigating the capability of advanced yield criteria (Barlat 1991 (Barlat et al., 1991a), Barlat 1994 (Barlat et al., 1997a), Barlat 1996 (Barlat et al., 1997b), Barlat 2000 (Barlat et al., 2003), and Cazacu et al. (Cazacu et al., 2006)) to describe the earing profile. These studies have led to the development of an accurate constitutive model, based on distortional hardening, for the calculation of the earing amplitude (Yoon et al., 2011a). Soare and Barlat (2010) reported that some of the recently proposed orthotropic yield functions obtained through the linear transformation method are in fact homogeneous polynomials, which simplifies FEM implementation of them. Chung at al. recently used different anisotropic yield function (Chung et al., 2011a) to model mechanical response of TWIP (twinning induced plasticity) automotive sheets including earing prediction. Also, Taherizadeh et al. (2010, 2011) reported, that prediction of earing can be significantly improved with the use of non-associated flow rule and mixed hardening model. Another approach was presented by Hallberg et al. (2007), who present a FEM simulation of cup drawing with constitutive model, which consider the formation of martensite in austenitic steels.

As mentioned above, cylindrical cups obtained by deep-drawing usually exhibit four ears. In the case of some materials having more specific anisotropy six or eight ears may develop during the drawing process as well. The experimental and theoretical studies have proved the existence of direct relationship between the number of ears and the variation of the anisotropy coefficient in the plane of the sheet metal. Namely, the occurrence of more than four ears can be predicted only by yield criteria that use at least eight material parameters associated to different planar directions in the identification procedure. In fact, such plasticity models have been developed mainly as a response to this challenge. An analysis of the number of mechanical parameters needed for describing the anisotropy and their influence on the predicted earing profile has been performed by Soare and Banabic (2008) using the Poly 8 yield criterion. The first yield criterion that uses more than eight mechanical characteristics in the identification procedure has been developed by Barlat et al. (2005). The capability of this model to predict the occurrence of six or eight ears in the drawing of cylindrical cups has been proved by Yoon et al. in a series of papers (Yoon et al., 2006, 2008, 2011b; Rousselier et al., 2009). Aiming at further improvement of the accuracy of the earing predictions Yoon et al. (2010) have also taken into account the evolution of the anisotropy during the sheet metal forming process. Soare and Barlat (2011) constructed ad-hoc extension of Yld2004 yield criterion in order to model finer detail in yield surface. Yoon et al. (2011a) have recently analyzed the capability of the series based yield criterion developed by Plunkett et al. (2008) to predict the earing profile of an aluminum alloy exhibiting strong anisotropy. Kim et al. (2008) used Srp2004-18p (Kim et al., 2007) plastic strain rate potential in order to predict more than four ears for AA2090-T3 aluminum alloy, whereas Park and Chung have used the Yld2000-2d vield criterion with non-associated flow rule (Park and Chung, 2012) to predict ears for AA2090-T3 and AA5042 aluminum sheets. The prediction reported is in quite good agreement with experiment.

Due to the increased processing capabilities of the computers, the solution of polycrystalline models has become possible. Recent results obtained using such models are presented in the papers (Raabe et al., 2005; Lela et al., 2009), as well as in the monographs published by Raabe et al. (2004) and Roters et al. (2010). Rousselier et al. (2009, 2010) use reduced texture of anisotropic hardening modified model to predict ears of 2090-T3 aluminum alloy. In such case the CPU time of the polycrystalline model is only five times larger than that with the simple von Mises model.

Chung and Shah (1992) and Chung et al. (1996) among others carried out simulations of a cylindrical cup drawing test using strain rate potentials suggested by Barlat et al. (1993). Rabahallah et al. (2009) also reported comparative results referring to the earing prediction with several strain-rate potentials (Srp 1993 (Barlat et al., 1993), Quantus (Arminjon et al., 1994), Srp 2004 (Barlat and Chung, 2005)).

Recently, a general plane-stress yield criterion (BBC2008) defined as an extension of the BBC 2005 model (Banabic et al., 2005) has been proposed by Comsa and Banabic (2008). Since in the identification procedure related to material characterization considering the new constitutive model more than eight mechanical parameters can be used, a more accurate description of the anisotropy is possible in comparison to the BBC 2005. The capability of the BBC2008 yield criterion to predict occurrence of more than four ears will be proved in the next sections of this paper by a numerical simulation of the cylindrical cup deep-drawing in the cases of two aluminum alloys and a fictitious material with highly anisotropic properties.

2. The BBC2008 yield criterion

The BBC2008 yield criterion is a plane-stress criterion, developed to describe the plastic behavior of highly orthotropic sheet metals (Comsa and Banabic, 2008). The yield surface defined by this model results from the implicit equation

$$\overline{\Phi}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, Y) := \overline{\sigma}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}) - Y = 0$$
(1)

where $\overline{\sigma}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}) \ge 0$ is the equivalent stress (see below), Y > 0 is the yield stress, while σ_{11}, σ_{22} , and $\sigma_{12} = \sigma_{21}$ are planar components of the stress tensor expressed in an orthonormal basis (1,2,3) superimposed upon the local axes of plastic orthotropy and the third unit vector of the local basis being always perpendicular to the mid-surface of the sheet metal (see Fig. 1).

The elasto-plastic constitutive model is completed with a definition of the respective state variables' evolution equations

$$d\sigma_{ij} = C_{ijkl} \left(d\varepsilon_{kl} - d\varepsilon_{kl}^{p} \right)$$

$$d\varepsilon_{ij}^{p} = \frac{\partial \overline{\Phi}}{\partial \sigma_{ij}} d\lambda$$

$$d\varepsilon_{eq}^{p} = \frac{\sigma_{kl} \frac{\partial \overline{\Phi}}{\partial \sigma_{kl}}}{M} d\lambda$$

$$Y = Y \left(\varepsilon_{eq}^{p} \right)$$
(2)

considering a decomposition of the total strain into elastic and plastic strain $d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}$, the associative flow rule and work hardening. Above, C_{ijkl} is the elastic stiffness tensor, λ is the plastic multiplier, ε_{eq}^{p} is the equivalent plastic strain and Y is yield stress.

In the particular case of the BBC2008 yield criterion, the equivalent stress is defined as follows (Comsa and Banabic, 2008):

$$\frac{\overline{\sigma}^{2k}}{W-1} = \sum_{r=1}^{s} \left\{ w^{r-1} \left\{ \left[L^{(r)} + M^{(r)} \right]^{2k} + \left[L^{(r)} - M^{(r)} \right]^{2k} \right\} \\
+ w^{s-r} \left\{ \left[M^{(r)} + N^{(r)} \right]^{2k} + \left[M^{(r)} - N^{(r)} \right]^{2k} \right\} \right\},$$

$$L^{(r)} = \varrho_{1}^{(r)} \sigma_{11} + \varrho_{2}^{(r)} \sigma_{22},$$

$$M^{(r)} = \sqrt{\left[m_{1}^{(r)} \sigma_{11} - m_{2}^{(r)} \sigma_{22} \right]^{2} + \left[m_{3}^{(r)} (\sigma_{12} + \sigma_{21}) \right]^{2}},$$

$$N^{(r)} = \sqrt{\left[n_{1}^{(r)} \sigma_{11} - n_{2}^{(r)} \sigma_{22} \right]^{2} + \left[n_{3}^{(r)} (\sigma_{12} + \sigma_{21}) \right]^{2}},$$

$$k, s \in \mathbb{N} \setminus \{0\}, \quad w = (3/2)^{1/s} > 1,$$

$$\varrho_{1}^{(r)}, \varrho_{2}^{(r)}, m_{1}^{(r)}, m_{2}^{(r)}, m_{3}^{(r)}, n_{1}^{(r)}, n_{3}^{(r)} \in \mathbb{R}.$$
(3)

The quantities denoted above as k, $\ell_1^{(r)}$, $\ell_2^{(r)}$, $m_1^{(r)}$, $m_2^{(r)}$, $m_3^{(r)}$, $n_1^{(r)}$,



Fig. 1. Material orthotropy and definition of (1,2,3) and (ξ,η,ς) coordinate systems.

 $n_2^{(r)}$ and $n_3^{(r)}$ (r = 1, ..., s) are material parameters. It is easily noticeable that the equivalent stress defined by Eq. (3) reduces to the isotropic formulation proposed by Barlat and Richmond (Barlat and Richmond, 1987) if

$$\begin{aligned} \boldsymbol{\ell}_1^{(r)} &= \boldsymbol{\ell}_2^{(r)} = m_1^{(r)} = m_2^{(r)} = m_3^{(r)} = n_1^{(r)} = n_2^{(r)} = n_3^{(r)} \\ &= 1/2, \quad r = 1, \dots, s. \end{aligned}$$

Under these circumstances, the value of the integer exponent k may be adopted according to the crystallographic structure of the sheet metal, as in Barlat and Richmond's model: k = 3 for BCC alloys, and k = 4 for FCC alloys.

The other material parameters involved in Eq. (3) are evaluated upon a corresponding identification procedure. Their number is $n_p = 8s$, wheres $\in \mathbb{N} \setminus \{0\}$ is the summation limit. Let n_e be the number of experimental values describing the plastic anisotropy of the sheet metal. If $n_e \ge 8$, the summation limit *s* must be chosen according to the constraint $s \le n_e/8$. When $n_e < 8$, the minimum value s = 1 must be adopted. In this case, the identification constraints obtained from experiments should be accompanied by at least $8 - n_e$ artificial conditions involving the material parameters. For example, if $n_e = 6$, one may enforce $m_1^{(1)} = n_1^{(1)}$ and $m_2^{(1)} = n_2^{(1)}$.

The crucial property of the yield surface regarding its parameters is its convexity. In Appendix A an extensive analysis on this topic is presented, which confirms that there is no constraint acting on the admissible values of the material parameters, included in the expression of the equivalent stress.

3. Identification of the BBC2008 yield criterion parameters

Due to expandable structure of the BBC2008 yield criterion many identification strategies can be devised. Comsa and Banabic developed a numerical procedure based on the minimisation of an error-function operating only with normalized yield stresses and *r*-coefficients obtained from uniaxial/biaxial tensile tests (Comsa and Banabic, 2008).

Let Y_{θ} be the yield stress predicted by the yield criterion in the case of uniaxial traction along a direction which is inclined by the angle θ from the rolling direction. The considered in-plane directions are denoted in Fig. 1 by axes ξ and 1, respectively. Correspondingly, we denote by 2 the direction transversal to the rolling direction and by η the direction, defined by the angle $\theta + 90^{\circ}$, while axis $3 = \varsigma$ denotes the direction perpendicular to the sheet plane.

The planar components of the stress tensor are in this case

$$\sigma_{11}|_{\theta} = Y_{\theta} \cos^2 \theta, \quad \sigma_{22}|_{\theta} = Y_{\theta} \sin^2 \theta, \quad \sigma_{12}|_{\theta} = \sigma_{21}|_{\theta}$$
$$= Y_{\theta} \sin \theta \cos \theta$$
(5)

After replacing them in Eq. (3), one gets the associated equivalent stress

$$\overline{\sigma}|_{\theta} = Y_{\theta}F_{\theta} \tag{6}$$

where F_{θ} is defined by the relationships

$$\frac{F_{\theta}^{2k}}{w-1} = \sum_{r=1}^{s} \left\{ w^{r-1} \left\{ \left[L_{\theta}^{(r)} + M_{\theta}^{(r)} \right]^{2k} + \left[L_{\theta}^{(r)} - M_{\theta}^{(r)} \right]^{2k} \right\} \\
+ w^{s-r} \left\{ \left[M_{\theta}^{(r)} + N_{\theta}^{(r)} \right]^{2k} + \left[M_{\theta}^{(r)} - N_{\theta}^{(r)} \right]^{2k} \right\} \right\} \\
L_{\theta}^{(r)} = \mathfrak{L}_{1}^{(r)} \cos^{2}\theta + \mathfrak{L}_{2}^{(r)} \sin^{2}\theta \\
M_{\theta}^{(r)} = \sqrt{\left[m_{1}^{(r)} \cos^{2}\theta - m_{2}^{(r)} \sin^{2}\theta \right]^{2} + \left[m_{3}^{(r)} \sin^{2}\theta \right]^{2}} \\
N_{\theta}^{(r)} = \sqrt{\left[n_{1}^{(r)} \cos^{2}\theta - n_{2}^{(r)} \sin^{2}\theta \right]^{2} + \left[n_{3}^{(r)} \sin^{2}\theta \right]^{2}} \tag{7}$$

Eqs. (1) and (6) lead to the following expression of the normalized uniaxial yield stress:

$$y_{\theta} = \frac{Y_{\theta}}{Y} = \frac{1}{F_{\theta}}$$
(8)

The *r*-coefficient corresponding to the uniaxial traction along ξ direction is defined by the formula

$$r_{\theta} = \frac{\mathrm{d}\varepsilon_{\eta\eta}^{p}}{\mathrm{d}\varepsilon_{33}^{p}} \tag{9}$$

where $d\epsilon_{\eta\eta}^p$ is the plastic strain-rate component associated to η direction and $d\epsilon_{33}^p$ is the through-thickness component of the same tensor. After some simple mathematical manipulations, Eq. (9) becomes

$$r_{\theta} = \frac{F_{\theta}}{G_{\theta}} - 1 \tag{10}$$

where G_{θ} is defined by the relationships

$$\begin{split} \frac{F_{\theta}^{2k-1}G_{\theta}}{w-1} &= \sum_{r=1}^{s} \left\{ w^{r-1} \Big[\widehat{L}_{\theta}^{(r)} + \widehat{M}_{\theta}^{(r)} \Big] \Big[L_{\theta}^{(r)} + M_{\theta}^{(r)} \Big]^{2k-1} \\ &+ w^{r-1} \Big[\widehat{L}_{\theta}^{(r)} - \widehat{M}_{\theta}^{(r)} \Big] \Big[L_{\theta}^{(r)} - M_{\theta}^{(r)} \Big]^{2k-1} \\ &+ w^{s-r} \Big[\widehat{M}_{\theta}^{(r)} + \widehat{N}_{\theta}^{(r)} \Big] \Big[M_{\theta}^{(r)} + N_{\theta}^{(r)} \Big]^{2k-1} \\ &+ w^{s-r} \Big[\widehat{M}_{\theta}^{(r)} - \widehat{N}_{\theta}^{(r)} \Big] \Big[M_{\theta}^{(r)} - N_{\theta}^{(r)} \Big]^{2k-1} \Big\} \end{split}$$

$$\hat{M}_{\theta}^{(r)} = \left[m_{1}^{(r)} - m_{2}^{(r)} \right] \left[m_{1}^{(r)} \cos^{2} \theta - m_{2}^{(r)} \sin^{2} \theta \right] / M_{\theta}^{(r)}
\hat{N}_{\theta}^{(r)} = \left[n_{1}^{(r)} - n_{2}^{(r)} \right] \left[n_{1}^{(r)} \cos^{2} \theta - n_{2}^{(r)} \sin^{2} \theta \right] / N_{\theta}^{(r)}$$
(11)

together with Eq. (7).

Let us denote by Y_b the yield stress predicted by the constitutive model in the case of in-plane uniform biaxial traction. The corresponding planar components of the stress tensor are

$$\sigma_{11}|_{b} = Y_{b}, \quad \sigma_{22}|_{b} = Y_{b}, \quad \sigma_{12}|_{b} = \sigma_{21}|_{b} = 0$$
(12)

After replacing them in Eq. (3), one gets the associated equivalent stress

$$\overline{\sigma}|_{b} = Y_{b}F_{b} \tag{13}$$

where F_b is defined by the relationships

$$\frac{F_b^{2k}}{w-1} = \sum_{r=1}^{s} \left\{ w^{r-1} \left\{ \left[L_b^{(r)} + M_b^{(r)} \right]^{2k} + \left[L_b^{(r)} - M_b^{(r)} \right]^{2k} \right\} \\
+ w^{s-r} \left\{ \left[M_b^{(r)} + N_b^{(r)} \right]^{2k} + \left[M_b^{(r)} - N_b^{(r)} \right]^{2k} \right\} \right\} \\
L_b^{(r)} = \ell_1^{(r)} + \ell_2^{(r)}, M_b^{(r)} = m_1^{(r)} - m_2^{(r)}, N_b^{(r)} = n_1^{(r)} - n_2^{(r)} \tag{14}$$

Eqs. (1) and (13) lead to the following expression of the normalized biaxial yield stress:

$$y_b = \frac{Y_b}{Y} = \frac{1}{F_b} \tag{15}$$

The *r*-coefficient corresponding to the biaxial traction is defined by the formula

$$r_b = \frac{\mathrm{d}\varepsilon_{22}^p}{\mathrm{d}\varepsilon_{11}^p} \tag{16}$$

After some simple mathematical manipulations, Eq. (16) becomes

$$r_b = \frac{F_b}{G_b} - 1 \tag{17}$$

where G_b is defined by the relationships

$$\begin{split} \frac{F_{b}^{2k-1}G_{b}}{w-1} &= \sum_{r=1}^{s} \left\{ w^{r-1} \Big[\widehat{L}_{b}^{(r)} + \widehat{M}_{b}^{(r)} \Big] \Big[L_{b}^{(r)} + M_{b}^{(r)} \Big]^{2k-1} \\ &+ w^{r-1} \Big[\widehat{L}_{b}^{(r)} - \widehat{M}_{b}^{(r)} \Big] \Big[L_{b}^{(r)} - M_{b}^{(r)} \Big]^{2k-1} \\ &+ w^{s-r} \Big[\widehat{M}_{b}^{(r)} + \widehat{N}_{b}^{(r)} \Big] \Big[M_{b}^{(r)} + N_{b}^{(r)} \Big]^{2k-1} \\ &+ w^{s-r} \Big[\widehat{M}_{b}^{(r)} - \widehat{N}_{b}^{(r)} \Big] \Big[M_{b}^{(r)} - N_{b}^{(r)} \Big]^{2k-1} \Big\} \end{split}$$

$$\widehat{L}_{b}^{(r)} = \ell_{1}^{(r)}, \quad \widehat{M}_{b}^{(r)} = m_{1}^{(r)}, \quad \widehat{N}_{b}^{(r)} = n_{1}^{(r)}$$
 (18)

together with Eq. (14).

An identification procedure that strictly enforces a large number of experimental constraints on the yield criterion would be inefficient in practical applications. The failure probability of such a strategy increases when the external restrictions become stronger. Taking into account this aspect, Comsa and Banabic, (2008) developed an identification procedure based on the minimization of the following error-function:

$$E\left[\begin{pmatrix} g_{1}^{(r)}, g_{2}^{(r)}, m_{1}^{(r)}, m_{2}^{(r)}, m_{3}^{(r)}, n_{1}^{(r)}, n_{2}^{(r)}, n_{3}^{(r)} \middle| r = 1, \dots, s \right] = \sum_{\theta_{j}} \left[\frac{y_{\theta_{j}}^{(\exp)}}{y_{\theta_{j}}} - 1 \right]^{2} + \sum_{\theta_{j}} \left[r_{\theta_{j}}^{(\exp)} - r_{\theta_{j}} \right]^{2} + \left[\frac{y_{b}^{(\exp)}}{y_{b}} - 1 \right]^{2} + \left[r_{b}^{(\exp)} - r_{b} \right]^{2}$$
(19)

where θ_j represents an individual element from a finite set of angles defining the orientation of the specimens used in the uniaxial tensile tests. One may notice that Eq. (19) describes a square-distance between the experimental and predicted values of the anisotropy characteristics. The minimization has been performed using a modified Levenberg–Marquardt algorithm, in which Jacobian of the error-function is evaluated numerically by forward-difference approximations.

The numerical tests performed by the authors have been focused on predicting the ear amplitude of a deep-drawn cup made from two aluminum alloys, AA5042-H2 and AA2090-T3. The numerical values have been taken from literature (Yoon et al., 2006, 2010).

Two versions of the BBC2008 yield criterion have been evaluated from the point of view of their performances. They include 8 and 16 material coefficients, respectively, and correspond to the smallest values of the summation limit (s = 1 and s = 2). The identification of the 16p BBC2008 model for both materials has been performed using all the mechanical parameters listed in Tables 1 and 2. In the case of the 8p BBC2008, the input data have been restricted to the values $y_{0}^{(exp)}$, $y_{45}^{(exp)}$, $y_{90}^{(exp)}$, $r_{0}^{(exp)}$, $r_{45}^{(exp)}$, $r_{90}^{(exp)}$, $r_{45}^{(exp)}$, $r_{90}^{(exp)}$, $r_{45}^{(exp)}$, $r_{90}^{(exp)}$, $r_{45}^{(exp)}$, $r_{90}^{(exp)}$, and $r_{b}^{(exp)}$. For comparison reasons the performance of the BBC2008 model is compared to the Yld2000-2d (Barlat et al., 2003), the Yld2004-18p (Barlat et al., 2005) and the CPB06ex2 (Plunkett et al., 2008) models, where possible.

3.1. Identification of the BBC2008 parameters for AA2090-T3

The input data, taken into identification procedure of the BBC2008 model parameters, are shown in Table 1.

Tables 2 and 3 contain the results of the identification procedure for the BBC2008 8p and the BBC2008 16p version, respectively.

From the point of flexibility of the model, it is interesting to show a comparison between the experimental data and the predictions of the BBC2008 model referring to the planar distribution of the normalized yield stress and *r*-coefficient in uniaxial tension. Fig. 2 shows the reproduction of *r*-coefficient and Fig. 3 show the reproduction of normalized yield stress. Also, the predictions of the

Table 1							
Anisotropy	/ character	istics of th	e AA2090-	T3 aluminu	um alloy (`	Yoon et al.	, 2006).

$y_{0^{\circ}}^{(\exp)}$	$y_{15^{\circ}}^{(\exp)}$	$y_{30^{\circ}}^{(\exp)}$	$y_{45^{\circ}}^{(\exp)}$	$y_{60^{\circ}}^{(\exp)}$	$y_{75^{\circ}}^{(\exp)}$	$y_{90^{\circ}}^{(\exp)}$	$y_b^{(exp)}$
1	0.961	0.910	0.811	0.810	0.882	0.910	1.035
$r_{0^{\circ}}^{(exp)}$	$r_{15^{\circ}}^{(exp)}$	$r_{30^{\circ}}^{(exp)}$	$r_{45^{\circ}}^{(exp)}$	$r_{60^{\circ}}^{(exp)}$	$r_{75^{\circ}}^{(exp)}$	$r_{00^{\circ}}^{(exp)}$	$r_{h}^{(exp)}$
0.212	0.327	0.692	1.577	1.039	0.538	0.692	0.670

Table 2	Та	bl	е	2
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Identified parameters of the 8p BBC2008 version – aluminum AA2090-T3.

k	s	w	$\ell_1^{(1)}$	$\ell_2^{(1)}$	$m_1^{(1)}$	$m_2^{(1)}$	$m_{3}^{(1)}$	$n_1^{(1)}$	$n_2^{(1)}$	$n_{3}^{(1)}$
4	1	1.5000	0.4499	0.5132	0.6303	0.6014	0.7273	0.1538	0.4794	0.4998

Table 3

Identified parameters of the 16p BBC2008 version - aluminum AA2090-T3.

k	S	w	$\mathfrak{l}_1^{(1)}$	$\mathfrak{l}_2^{(1)}$	$m_1^{(1)}$	$m_2^{(1)}$	m ₃ ⁽¹⁾	$n_1^{(1)}$	$n_2^{(1)}$
$\frac{4}{n_{2}^{(1)}}$	2	1.2247 l ⁽²⁾	0.1309 $l_{2}^{(2)}$	$0.6217 m_{\star}^{(2)}$	$0.7834 m_{2}^{(2)}$	$\frac{0.6604}{m_2^{(2)}}$	0.79×10^{-4} $n_{\star}^{(2)}$	0.1110 $n_{2}^{(2)}$	$\frac{0.0482}{n_2^{(2)}}$
0.3075		1.0339	-0.0720	1.13×10^{-4}	0.77×10^{-4}	0.5380	0.0558	1.0186	0.7781

Yld2002-2d, the Yld2004-18p and the CPB06ex2 models are shown in the same plots for comparison reason.

The 8p version of the BBC2008 model is able to reproduce exactly all the input data used in the identification. This situation is a consequence of the fact that the yield surface is subjected to fewer constraints when only eight experimental values are used. Due to this fact, its performances are far from being satisfactory when analyzing the predictions of *r*-coefficient at directions 15°, 30°, 60° and 75° (Fig. 2). In contrast, the planar distribution of *r*-coefficients is closely followed by the 16p version of the BBC2008 model. In fact, with the detailed analyze of the predicted *r*-coefficients we can conclude, that for observed material the BBC2008 16p predicts *r*-coefficients even better than the Yld2004-18p and the CPB06ex2 models.

Another task of the anisotropic yield criterion is to predict planar distribution of normalized uniaxial yield stress. From observing Fig. 3 one may notice that the uniaxial yield stress prediction of the 8p version of the BBC2008 is similar to the prediction of the Yld2002-2d model, as expected. The 16p version of the BBC2008 again outperforms the prediction of 8p version and is from qualitative point of view similar to the predictions of the Yld2004-18p and the CPB06ex2 prediction.

Fig. 4 shows the shape of the normalized yield surface for the 8p and 16p versions of the BBC2008 model.

3.2. Identification of the BBC2008 parameters for AA5042-H2

In the case of AA5042-H2 material there are more available input data that can be taken into identification procedure of the BBC model. In the paper of Yoon et al. (2010) beside Lankford values and initial yield stress ratios also yield stress ratios at 10% of



Fig. 2. Planar distribution of *r*-coefficient predicted by the BBC2008 model for the AA2090-T3 aluminum alloy.

equivalent plastic deformation are measured. The measured data for AA5042-H2 aluminum alloy are shown in Table 4, whereas superscript (exp 0.1) denotes, that data have been measured at 10% of equivalent plastic strain.

The same procedure as in the case of AA2090-T3 aluminum alloy has been followed to obtain the BBC2008 model parameters for AA5042-H2 aluminum alloy. Two sets of parameters were identified independently as consequence of two sets if input data for AA5042-H2 (initial yield stress ratios and yield stress ratios at 10% of equivalent plastic deformation). The identified parameters are shown in Tables 5–8.

Again, as in the case of AA2090-T3 aluminum alloy, it is interesting to show comparisons between the experimental data and the predictions of the BBC2008 model referring to the planar distribution of the normalized yield stress and *r*-coefficient in uniaxial tension.

Figs. 5 and 6 show the predictions of planar distribution of the normalized yield stress and *r*-coefficient in uniaxial tension provided by the BBC2008 model for the initial yield stress ratios. For the sake of comparison, also the prediction, made by the CPB06ex2 model are included.

Again, the agreement between experimental and predicted Lankford values and yield stress ratios is better in the case of 16p version of BBC2008, as expected. The 16p version of BBC2008 fits the Lankford values even better as CPB06ex2 model. The prediction of stress ratios for 16p version of BBC2008 is in the same level as the prediction of CPB06ex2 model.

The identification was performed also on second set of data, thus on the same Lankford values but considering yield stress ratios at 10% of equivalent plastic strain. Agreement between experimental and predicted Lankford values is shown on Fig. 7, whereas Fig. 8 shows the agreement between planar yield stress ratios distribution.



Fig. 3. Planar distribution of normalized uniaxial yield stress predicted by the BBC2008 model for the AA2090-T3 aluminum alloy.



Fig. 4. Normalized yield surface predicted by the BBC2008 model for the AA2090-T3 aluminum alloy.

Again, 16p version better fits the experimental data as 8p version of the BBC2008 yield criterion, as expected. In comparison with the CPB06ex2 model 16p version of the BBC2008 better predict *r*-values whereas the prediction of yield stress distribution is at least qualitatively slightly better with the CPB06ex2 model. Since according to the experimental and theoretical observations formulated by Yoon et al. (2006) and Chung et al. (2011b), the earing profile of a drawn cup is mainly determined by the *r*-distribution, the slight disagreement in yield stress distribution is acceptable. Fig. 9 shows the shape of the normalized yield surface for 8p and 16p versions of the BBC2008 model for AA5042-H2 aluminum alloy.

The difference between both models in biaxial region is mainly due to the fact that in the identification procedure of the CPB06ex2 model the biaxial *r*-value was not considered.

4. Implementation of the BBC2008 yield criterion in ABAQUS

The above presented constitutive model has been implemented in a general purpose finite element code ABAQUS via VUMAT subroutine (ABAQUS Version 6.8, 2006). For the integration of the constitutive model the NICE (*Next Increment Corrects Error*) explicit integration scheme, developed recently by some co-authors, is used. Its task is to find a proper increment of the plastic multiplier $\Delta\lambda$ from a given total strain increment $\Delta\varepsilon_{ij}$. The basic ideas of the NICE scheme are presented in Halilovic et al. (2009), whereas in Vrh et al. (2010) its theoretical background is adequately elaborated. The comparison studies and the proof given in the appendix of the latter paper show, that accuracy of the new scheme is comparable to the accuracy of the backward-Euler scheme, while it is

Anisotropy characteristics of the AA5042-H2 aluminum alloy (Yoon et al., 2010).

Table 4

$y_{0}^{(\exp)}$	$y_{15^{\circ}}^{(exp)}$	$y_{30}^{(exp)}$	$y_{45}^{(exp)}$	$y_{60^{\circ}}^{(exp)}$	$y_{75^{\circ}}^{(exp)}$	$y_{90}^{(exp)}$	$y_{h}^{(\exp)}$
0.902	0.914	0.923	0.924	0.932	0.946	0.940	1
$y_{0^{\circ}}^{(\exp 0.1)}$	$y_{15^{\circ}}^{(\exp 0.1)}$	$y_{30^{\circ}}^{(\exp 0.1)}$	$y_{45^{\circ}}^{(\exp 0.1)}$	$y_{60^{\circ}}^{(\exp 0.1)}$	$y_{75^{\circ}}^{(\exp 0.1)}$	$y_{90^{\circ}}^{(\exp 0.1)}$	$y_h^{(\exp 0.1)}$
0.929	0.931	0.921	0.911	0.911	0.929	0.935	1
$r_{0^{\circ}}^{(\exp)}$	$r_{15^{\circ}}^{(exp)}$	$r_{30^{\circ}}^{(exp)}$	$r_{45^{\circ}}^{(exp)}$	$r_{60^{\circ}}^{(exp)}$	$r_{75^{\circ}}^{(exp)}$	$r_{90^{\circ}}^{(exp)}$	$r_{h}^{(exp)}$
0.354	0.239	0.640	1.069	1.279	1.224	1.396	0.991

Table 5

Identified parameters of the 8p BBC2008 version – aluminum AA5042-H2 (initial yield stress ratios).

k	s	w	l ⁽¹⁾	$\ell_{2}^{(1)}$	$m_{1}^{(1)}$	$m_{2}^{(1)}$	$m_{2}^{(1)}$
4	1	1.5	0.6473	0.3478	0.5540	0.5914	0.5813
$n_1^{(1)}$	$n_{2}^{(1)}$	$n_{3}^{(1)}$					
0.2796	0.5388	0.4866					

computationally far more (up to ten times) efficient in explicit dynamics simulations.

An implementation of the constitutive model via user subroutine requires integration of the constitutive model along a known strain path, which is mathematically reflected in known total strain increments $\Delta \varepsilon_{ij}$. Although a deduction of the NICE integration scheme is general, its implementation for shell applications needs a particular care. Namely, in order to satisfy the zero normal stress condition during the whole integration a through-thickness strain increment has to be adequately chosen in each integration step.

4.1. Treatment of zero normal stress constraint in shell applications

Because the proposed yield criterion can be used in plane-stress applications only, the through-thickness strain increment $\Delta \varepsilon_{33}$ cannot be determined from a displacement field directly, but indirectly. A general approach to the through-thickness strain increment calculation is derived in Vrh et al. (2010). In classical elasto-plastic plane stress applications the user of the NICE scheme can calculate the through-thickness strain increment with the following simple equation

$$\Delta \varepsilon_{33} = -\frac{C_{33kl} \Delta \varepsilon_{kl}^* - \left(\Phi + \frac{\partial \Phi}{\partial \sigma_{ij}} C_{ijkl} \Delta \varepsilon_{kl}^*\right) \Theta \beta}{\left(C_{3333} - \Theta^2 \beta\right)}$$
(20)

where

$$\beta = \left(\frac{\partial \Phi}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial \Phi}{\partial \sigma_{kl}} - \frac{\partial \Phi}{\partial Y} \frac{dY}{d\varepsilon_{eq}^p} \frac{\partial \Phi}{\partial \sigma_{ij}} \sigma_{ij}}{Y}\right)^{-1}, \ \Theta = \frac{\partial \Phi}{\partial \sigma_{ij}} C_{ij33} ,$$

$$\varepsilon_{ij}^* = \begin{cases} 0 & ; \ i = j = 3 \\ \varepsilon_{ij} & ; \ \text{otherwise} \end{cases}$$
(21)

and C_{ijkl} is the elasticity tensor, $Y = Y(\varepsilon_{eq}^p)$ is the yield stress as a function of the equivalent plastic strain ε_{eq}^p and $\Phi(\sigma_{ij}, Y)$ is the plastic potential.

With reference to here discussed BBC2008 yield criterion (see Eqs. (1) and (3)) an alternative form will be used in the numerical implementation for the plastic potential (see Eq. (31)). Either of these two representations $\overline{\Phi}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, Y) = 0$ is actually a plane stress representation of the respective yield criterion $\Phi(\sigma_{ij}, Y) = 0$, which is expressed in terms of arbitrary stress state components σ_{ij} . Thus, the following relationship holds

$$\overline{\Phi}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, Y) = \Phi(\sigma_{ij}, Y; \sigma_{i3} = \sigma_{3j} = 0)$$
(22)

Table 6

Identified parameters of the 16p BBC2008 version - aluminum AA5042-H2 (initial yield stress ratios).

k	S	w	$\mathfrak{l}_1^{(1)}$	$\ell_2^{(1)}$	$m_1^{(1)}$	$m_2^{(1)}$	$m_3^{(1)}$
4	2	1.2247	0.3527	-0.7187	0	0	-0.8769
$n_1^{(1)}$	$n_2^{(1)}$	$n_{3}^{(1)}$	$l_1^{(2)}$	$l_{2}^{(2)}$	$m_1^{(2)}$	$m_2^{(2)}$	$m_3^{(2)}$
-0.4479	-0.0714	-0.2061	0.7275	0.3431	-0.5720	$-\bar{0}.6217$	0.5675
$n_1^{(2)}$	$n_2^{(2)}$	$n_{3}^{(2)}$					
-0.2992	-0.6359	0.					

Identified parameters of the 8p BBC2008 version	n — aluminum AA5042-H2 (yield stress ratios at	0.1 of equivalent plastic deformat	ion).
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$\frac{k}{n_1^{(1)}}$	$s \\ 1 \\ n_2^{(1)}$	$w \\ 1.5 \\ n_2^{(1)}$	${rak k}_1^{(1)} - 0.6124$	$\ell_2^{(1)} - 0.3847$	$m_1^{(1)} - 0.5490$	$m_2^{(1)} - 0.5776$	$m_3^{(1)}$ 0.5990
-0.3110	-0.5538	-0.4826					

While the functional form of $\overline{\Phi}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, Y)$ is known explicitly this is not the case for function $\Phi(\sigma_{ij}, Y)$. Yet, in the equations above, as well as in the corresponding evolution equations of the problem state variables which are in (27) given in their incremental form, derivatives of function $\Phi(\sigma_{ij}, Y)$ are required. In overcoming this enigma we shall first take the properties of partial differentiation into account, which yields when the plane stress state is considered

$$\frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{\partial \overline{\Phi}}{\partial \sigma_{ij}}; \quad i, j \in \{1, 2\} \quad \land \quad \sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \tag{23}$$

The plane state stress is also characterized by $d\epsilon_{13}^p = d\epsilon_{23}^p = 0$, which yields when the corresponding evolution Equation (27) are considered

$$\frac{\partial \Phi}{\partial \sigma_{i3}} = \frac{\partial \Phi}{\partial \sigma_{i3}} = 0; \quad i \in \{1, 2\} \quad \land \quad \sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$
(24)

The remaining derivative $\partial \Phi / \partial \sigma_{33}$ may be computed from the plastic incompressibility condition $d\varepsilon_{kk}^p = 0$, which yields in accordance with (27)

$$\frac{\partial \Phi}{\partial \sigma_{33}} = -\left(\frac{\partial \Phi}{\partial \sigma_{11}} + \frac{\partial \Phi}{\partial \sigma_{22}}\right) = -\left(\frac{\partial \overline{\Phi}}{\partial \sigma_{11}} + \frac{\partial \overline{\Phi}}{\partial \sigma_{22}}\right) ; \ \sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$
(25)

4.2. Integration of the BBC2008 constitutive model with the NICE scheme

Assuming, that in each increment the total strain increments $\Delta \varepsilon_{ij}$ except $\Delta \varepsilon_{33}$, which can be calculated by considering (20), are available from the computed increments of the displacement field, the stress state at the end of the increment must be calculated using the following system of algebraic equations:

$$\Phi(\sigma_{ij}, Y) = 0 \tag{26}$$

$$\Delta \sigma_{ij} = C_{ijkl} \left(\Delta \varepsilon_{kl} - \Delta \varepsilon_{kl}^{p} \right)$$

$$\Delta \varepsilon_{ij}^{p} = \frac{\partial \Phi}{\partial \sigma_{ij}} \Delta \lambda$$

$$\Delta \varepsilon_{eq}^{p} = \frac{\sigma_{kl} \frac{\partial \Phi}{\partial \sigma_{kl}}}{Y} \Delta \lambda$$

$$\Delta Y = \frac{\partial Y}{\partial \varepsilon_{eq}} \Delta \varepsilon_{eq}^{p}$$
(27)

The above system defines that the incremental evolution of the problem state variables (27) is constrained by the consistency condition (26).

Table 8

Identified parameters of the 16p BBC2008 version – aluminum AA5042-H2 (yield stress ratios at 0.1 of equivalent plastic deformation).

k	S	w	$\mathfrak{l}_1^{(1)}$	$\ell_2^{(1)}$	$m_1^{(1)}$	$m_2^{(1)}$	$m_3^{(1)}$
4	2	1.2247	0.3482	-0.7058	-0.0167	0.2308	0.8936
$n_1^{(1)}$	$n_2^{(1)}$	$n_{3}^{(1)}$	$l_1^{(2)}$	$l_2^{(2)}$	$m_1^{(2)}$	$m_2^{(2)}$	$m_{3}^{(2)}$
-0.4052	-0.1255	-0.2100	0.7181	0.3516	-0.5418	-0.5906	0.5994
$n_1^{(2)}$	$n_2^{(2)}$	$n_{3}^{(2)}$					
-0.3338	-0.6684	0.					

The major concern regarding a numerical solution of any boundary value problem involving plastic deformation is how to ensure the consistency condition (26) is respected on the entire integration path (Halilovic et al., 2009). According to the NICE scheme, this is achieved by expanding the consistency condition into Taylor power series expansion, where higher-order differentials are neglected. The numerical scheme is thus based, provided the values of the state variables are known at the beginning of the considered increment, on imposing

$$\Phi + d\Phi = 0 \tag{28}$$

to be fulfilled in the considered increment, which leads to

$$\Phi + \frac{\partial \Phi}{\partial \sigma_{ij}} \Delta \sigma_{ij} + \frac{\partial \Phi}{\partial Y} \Delta Y = 0$$
⁽²⁹⁾

With regard to the forward-Euler approach which uses a differential form of the consistency condition, i.e. $d\Phi = 0$, our approach considers the additional term Φ . Though this term should be zero, because it represents a function whose value should obey the consistency condition $\Phi = 0$, numerically this is usually not true. This small difference between the two explicit schemes, the NICE and the forward-Euler, is the key reason for a considerable improvement which is proven by the NICE scheme with respect to the forward-Euler scheme both in accuracy and stability of the numerical integration.

From the examination of Eqs. (26) and (27) it follows that the incremental solution depends on a consistently determined increment of the plastic multiplier $\Delta \lambda$. In the case of adopting the NICE scheme with the consistency condition considered in its expanded form (29) the following expression is obtained for the increment of the plastic multiplier

$$\Delta\lambda = \frac{\Phi + \frac{\partial\Phi}{\partial\sigma_{ij}}C_{ijkl}\Delta\varepsilon_{kl}}{\frac{\partial\Phi}{\partial\sigma_{ij}}C_{ijkl}\frac{\partial\Phi}{\partial\sigma_{kl}} - \frac{\partial\Phi}{\partial Y}\frac{dY}{d\varepsilon_{eq}}\frac{\sigma_{kl}\frac{\partial\Phi}{\partial\sigma_{kl}}}{Y}}{Y}$$
(30)

In the cases, where yield function exhibits first order homogeneity, the second term of the denominator in Eq. (30) can be



Fig. 5. Planar distribution of *r*-coefficient predicted by the BBC2008 model for the AA5042-H2 aluminum alloy (initial yield stress ratios).

Table 7



Fig. 6. Planar distribution of yield stress ratios predicted by the BBC2008 model for the AA5042-H2 aluminum alloy (initial yield stress ratios).

simplified to instantaneous hardening slope. When the increment of the plastic multiplier $\Delta \lambda$ is calculated, the incremental solution of the elasto-plastic model, defined by the evolution equation (27), can be calculated explicitly from evolution equations.

The numerical integration procedure, see Eqs. (27) and (30), requires a calculation of the yield function $\Phi(\sigma_{ij}, Y)$ and its respective derivatives in every increment. For implementation purposes, the yield function (1) can be rewritten in the following form:

$$\overline{\Phi} = \frac{\overline{\sigma}^{2k}(\sigma_{ij})}{Y^{2k}} - 1 = \frac{\widehat{\sigma}(\sigma_{ij})}{Y^{2k}} - 1 = 0$$
(31)

The derivatives used in the numerical scheme are now computed by considering relations (22)–(25) established between the two functions, $\Phi(\sigma_{ij}, Y)$ and $\overline{\Phi}(\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}, Y)$, which yields:

$$\frac{\partial\Phi}{\partial Y} = \frac{\partial\overline{\Phi}}{\partial Y} = -2k\frac{\widehat{\sigma}}{Y^{2k+1}} ; \quad \frac{\partial\Phi}{\partial\overline{\sigma}} = \frac{\partial\overline{\Phi}}{\partial\overline{\sigma}} = \frac{1}{Y^{2k}}$$

$$\frac{\partial\Phi}{\partial\sigma_{ij}} = \frac{\partial\overline{\Phi}}{\partial\sigma_{ij}} = \frac{\partial\overline{\Phi}}{\partial\overline{\sigma}} \sum_{r=1}^{S} \left[\frac{\partial\widehat{\sigma}}{\partial L^{(r)}} \frac{\partial L^{(r)}}{\partial\sigma_{ij}} + \frac{\partial\widehat{\sigma}}{\partial M^{(r)}} \frac{\partial M^{(r)}}{\partial\sigma_{ij}} + \frac{\partial\widehat{\sigma}}{\partial N^{(r)}} \frac{\partial N^{(r)}}{\partial\sigma_{ij}} \right] ; \quad i,j \in \{1,2\}$$

$$\frac{\partial\Phi}{\partial\sigma_{13}} = \frac{\partial\Phi}{\partial\sigma_{23}} = 0 ; \quad \frac{\partial\Phi}{\partial\sigma_{33}} = -\left(\frac{\partial\Phi}{\partial\sigma_{11}} + \frac{\partial\Phi}{\partial\sigma_{22}}\right)$$
(32)

where



Fig. 7. Planar distribution of *r*-coefficient predicted by the BBC2008 model for the AA5042-H2 aluminum alloy (yield stress ratios at 0.1 of equivalent plastic strain).

$$\begin{split} \frac{\partial \widehat{\sigma}}{\partial L^{(r)}} &= 2k(w-1)w^{r-1} \left[\left(L^{(r)} + M^{(r)} \right)^{2k-1} + \left(L^{(r)} - M^{(r)} \right)^{2k-1} \right] \\ \frac{\partial \widehat{\sigma}}{\partial N^{(r)}} &= 2k(w-1)w^{s-r} \left[\left(M^{(r)} + N^{(r)} \right)^{2k-1} - \left(M^{(r)} - N^{(r)} \right)^{2k-1} \right] \\ \frac{\partial \widehat{\sigma}}{\partial M^{(r)}} &= 2k(w-1) \left\{ w^{r-1} \left[\left(L^{(r)} + M^{(r)} \right)^{2k-1} - \left(L^{(r)} - M^{(r)} \right)^{2k-1} \right] \\ &+ w^{s-r} \left[\left(M^{(r)} + N^{(r)} \right)^{2k-1} + \left(M^{(r)} - N^{(r)} \right)^{2k-1} \right] \right\} \end{split}$$
(33)

and

$$\begin{split} \frac{\partial L^{(r)}}{\partial \sigma} &= \left\{ \ell_{1}^{(r)}, \ell_{2}^{(r)}, -\left(\ell_{1}^{(r)} + \ell_{2}^{(r)}\right), 0 \right\} \\ \frac{\partial M^{(r)}}{\partial \sigma} &= \frac{1}{M^{(r)}} \left\{ \tilde{M}^{(r)} m_{1}^{(r)}, -\tilde{M}^{(r)} m_{2}^{(r)}, \tilde{M}^{(r)} \left(m_{2}^{(r)} - m_{1}^{(r)} \right), m_{3}^{(r)} 2\sigma_{12} \right\} \\ \frac{\partial N^{(r)}}{\partial \sigma} &= \frac{1}{N^{(r)}} \left\{ \tilde{N}^{(r)} n_{1}^{(r)}, -\tilde{N}^{(r)} n_{2}^{(r)}, \tilde{N}^{(r)} \left(n_{2}^{(r)} - n_{1}^{(r)} \right), n_{3}^{(r)} 2\sigma_{12} \right\} \\ \tilde{M}^{(r)} &= m_{1}^{(r)} (\sigma_{11} - \sigma_{33}) - m_{2}^{(r)} (\sigma_{22} - \sigma_{33}) \\ \tilde{N}^{(r)} &= n_{1}^{(r)} (\sigma_{11} - \sigma_{33}) - n_{2}^{(r)} (\sigma_{22} - \sigma_{33}) \end{split}$$
(34)

Equation (34) are written, due to simplicity, in Voigt notation, where $\frac{\partial}{\partial \sigma} = \left\{ \frac{\partial}{\partial \sigma_{11}}, \frac{\partial}{\partial \sigma_{22}}, \frac{\partial}{\partial \sigma_{33}}, \frac{\partial}{\partial \sigma_{12}} \right\}$.

Let us remind that according to the explicit approach all the state variables appearing in the above equations and expressions are written at the beginning of the considered increment. Once the increment of the plastic multiplier $\Delta\lambda$ is calculated, the respective increments of the other state variables can be readily calculated by considering (27). The corresponding flow chart of the applied incremental procedure used for the integration of the constitutive model is presented in Fig. 10.

The described procedure can be used to integrate any of elastoplastic constitutive law.

5. Earing prediction using the BBC2008 yield criterion

The constitutive model resulted by considering the associated flow theory of plasticity and assuming the BBC2008 yield criterion has been thoroughly discussed in the previous sections. Issues, such as material parameters characterization and numerical implementation which are crucial for a physically objective computational analysis, have been addressed in the previous sections. In this section we will validate, in order to identify capability of the constitutive model as well as the effectiveness of the conceived



Fig. 8. Planar distribution of yield stress ratios predicted by the BBC2008 model for the AA5042-H2 aluminum alloy (yield stress ratios at 0.1 of equivalent plastic strain).



Fig. 9. Normalized yield surfaces predicted by the BBC2008 and CPB06ex2 models for the AA5042-H2 aluminum alloy; a) initial yield ratios, b) yield ratios at 0.1 of equivalent plastic strains.

numerical approach in the analysis of deep drawing processes, the results obtained from a corresponding computer simulation of the sheet metal forming process. In particular, two cases of deep drawing of a round cup made respectively from two aluminum alloys, AA2090-T3 and AA5042-H2, will be considered. In order to show the flexibility of the BBC2008 model an additional subsection will be devoted to a highly anisotropic fictitious material, which yields, according to the theory, more than eight ears in cup drawing.

In the modeling of the considered round cup deep drawing only a quarter section of the cup with the corresponding symmetry boundary conditions applied is analyzed, due to orthotropic material properties. A total of 2560 shell elements with reduced integration (ABAQUS S4R) and 21 section points through the sheet thickness are used for the simulation. The implementation of the



Fig. 10. Integration of the constitutive model – incremental procedure.

BBC2008 model with the respective parameters identified for each of the sheet metals considered is enabled via user material sub-routine VUMAT.

5.1. Earing prediction for AA2090-T3

In this subsection the BBC2008 model is validated on aluminum AA2090-T3. The case considered refers to the round cup drawing experiment and the corresponding computer simulation with the Yld2004 model, presented in Yoon et al. (2006). The tool geometry is specified as follows: blank diameter is 158.76 mm, die opening diameter is 101.48 mm, punch diameter is 97.46 mm, die-profile radius and punch-profile radius are 12.70 mm both. The initial thickness of the blank is 1.6 mm and the holder force is of magnitude 22.2 kN. The contact conditions are governed by the Coulomb friction model and magnitude 0.1 being assumed for the friction coefficient. The material hardening is modeled with the following stress–strain curve $Y = 646(0.025 + \varepsilon_{eq}^{p})^{0.227}$. The material anisotropy characterization is described in Section 3, where also the identified BBC2008 parameters are tabulated.

Fig. 11 displays the plots of the numerically predicted earing profile (for 8p and 16p BBC2008 version) and the experimentally established one (Yoon et al., 2006). For the sake of comparison, also the calculated earing profile with the Yld2004 model (Yoon et al., 2006) is included.

The simulated predictions with the BBC2008 model are in good agreement with the results of the Yld2004 model and also with the experiment (Yoon et al., 2006). As expected, the simulation considering the 8p BBC2008 version was unable to predict six ears, which were observed experimentally. On the contrary, the 16p BBC2008 version simulation predicts the number of ears (six) and their location correctly, and at least qualitatively, the results are in good agreement with the experiment.

5.2. Earing prediction for AA5042-H2

The second case considered is a deep drawing simulation of aluminum AA5042-H2 with the applied process data following those of the case, investigated in Yoon et al. (2010). The tool geometry is specified as follows: blank diameter is 76.07 mm, die opening diameter is 46.74 mm, punch diameter is 45.72 mm, dieprofile radius and punch-profile radius are 2.28 mm both. The initial thickness of the blank is 0.274 mm and the holder force is of magnitude 10 kN. The punch displacement is set to be large enough to pull the whole blank into the die. The friction between the tools



Fig. 11. Experimental vs. numerical earing prediction for aluminum AA2090-T3.



Fig. 12. Experimental vs. numerical earing prediction for AA5042-H2 aluminum alloy (initial yield stress).

and deforming sheet metal is considered in accordance with the Coulomb model, the coefficient of friction being 0.008 for all surfaces in contact. In this case the Voce hardening law $Y = 404.16 - 107.17 \ e^{-18.416} \ e_{eq}^{p}$ is assumed to model the work hardening behavior of the sheet metal.

In Fig. 12 the numerically predicted earing profile and the experimentally established one (Yoon et al., 2010) are plotted. The earing profile is defined by the height of the deep drawn cup $h(\theta)$ along 1/4-circumference of the cup, as a function of angle θ (angle $\theta = 0^{\circ}$ denotes the rolling direction). For the sake of comparison, also the calculated earing profile with the CPB06ex2 model (Yoon et al., 2010) is included. Both predictions are made with initial yield stress ratios being considered in parameters identification.

Another prediction can be made with model parameters, which were identified considering yield stress ratios at 0.1 of equivalent plastic strain. Such prediction is shown in Fig. 13, where also prediction of the CPB06ex2 model (Yoon et al., 2010) are included, considering the same material data.

Regarding the simulated earing profile plots in Figs. 12 and 13 we can state that the earing profile prediction with the BBC2008 model is in rather good qualitative agreement with experiments, especially in range 45°-90°. As expected, the 8p version of BBC2008 is not capable of predicting eight ears; it predicts only four ears, but cup height in range 45°-90° are predicted quite accurately. The 16p version of the BBC2008 clearly improves the prediction of the 8p version of the BBC2008 model in such a way, that it creates additional four ears. In general, it can be concluded, that the earing prediction of the BBC2008 model is comparable to the prediction of the CPB06ex2 model, proposed by Yoon et al. (2010). The predicted ears are in the same location and the agreement with the experimental results is from the qualitative and quantitative point of view in the same level. It is also interesting to observe, that the consideration of vield stress ratios at 0.1 of equivalent plastic strain does not considerably affect prediction of the earing in the case of the BBC2008 model. This statement is not valid for the CPB06ex2 model, where the consideration of different yield stress ratios gives quite different prediction of earing, especially at 90°.



Fig. 13. Experimental vs. numerical earing prediction for AA5042-H2 aluminum alloy (yield stress ratios at 0.1 of equivalent plastic strain).

Anisotropy	characteristics	of the	fictitious	material

$y_{0^{\circ}}^{(\exp)}$	$y_{15^{\circ}}^{(\exp)}$	$y_{30^{\circ}}^{(\exp)}$	$y_{45^{\circ}}^{(\exp)}$	$y_{60^{\circ}}^{(\exp)}$	$y_{75^{\circ}}^{(\exp)}$	$y_{90^{\circ}}^{(\exp)}$	$y_{b}^{(\exp)}$
1.0000	1.027	1.067	1.088	1.067	1.027	1.000	1.050
$r_{0^{\circ}}^{(\exp)}$	$r_{15^{\circ}}^{(exp)}$	$r_{30^{\circ}}^{(exp)}$	$r_{45^{\circ}}^{(exp)}$	$r_{60^{\circ}}^{(exp)}$	$r_{75^{\circ}}^{(exp)}$	$r_{90^{\circ}}^{(exp)}$	
1.500	1.007	1.625	1.458	1.853	1.057	1.502	

5.3. Example of predicting more than eight ears using fictitious material

In order to show the flexibility of the BBC2008 model in modeling of highly anisotropic sheet metal response we follow, in the sequel, the approach, presented in the paper of Yoon et al. (2006) and introduce a highly anisotropic fictitious material with material data generated. The respective normalized yield stresses and *r*-values are tabulated in Table 9. The data are generated in such a way, that according to the well recognized relation between *r*-values and ears (Yoon et al., 2006) they yield twelve ears. The key issue regarding this example is thus associated with the question whether the BBC2008 model is also capable of predicting twelve ears in cup simulation.

Parameters of the BBC2008 model were identified in the same way, as in the case of AA2090-T3 and AA5042-H2 aluminum alloys, using the procedure described in Section 3. Since 8p version was unable to reproduce the generated material data, only 16p version of BBC2008 model was used to model the fictitious material. The identified parameters are tabulated in Table 10.

The agreement between generated material data and planar distribution of yield stress ratios and *r*-values, predicted by the BBC2008 model, is shown in Figs. 14 and 15, respectively.

By observing Figs. 14 and 15 it can be concluded, that 16p version of the BBC2008 model is capable of reproducing the generated data very accurately. In Fig. 16 normalized yield surface of the BBC2008 16p for the fictitious material is shown.

Finally, it is interesting to observe the earing prediction in cup drawing simulation for the investigated fictitious material. The case considered here is a deep drawing simulation of a round cup with the applied process data following those of the AA5042-H2 case (see previous Subsection). Fig. 17 presents the respective cup simulation results – the equivalent plastic strain distribution and the cup height variation along the circumference. From the latter the appearance of twelve ears can be clearly observed.

The above example proves that despite of its simple formulation the BBC2008 model containing 16 parameters for anisotropy description only is flexible enough to predict more than eight ears in cup drawing simulation, if needed.

6. Conclusion

In this paper the authors have shown an approach to the modeling of plastic behavior in highly anisotropic sheet metals which is followed, in order to ensure computationally efficient and physically objective computer simulation, by the procedures required in its numerical implementation. Because of the finite series form that can be, depending on the amount of available



Fig. 14. Planar distribution of *r*-coefficient predicted by the BBC2008 model for the fictitious material.



Fig. 15. Planar distribution of yield stress ratios predicted by the BBC2008 model for the fictitious material.



Fig. 16. Normalized yield surface predicted by the 16p version of the BBC2008 for the fictitious material.

Table 10	
Identified parameters of the 16p BBC2008 ve	ersion.

		•							
k	S	w	$\ell_1^{(1)}$	$\ell_2^{(1)}$	$m_1^{(1)}$	$m_2^{(1)}$	$m_{3}^{(1)}$	$n_1^{(1)}$	n ₂ ⁽¹⁾
4	2	1.2247	0.6228	-0.6048	0.4973	0.5095	0.0000	0.0000	0.0000
$n_3^{(1)}$		$l_1^{(2)}$	$l_2^{(2)}$	$m_1^{(2)}$	$m_2^{(2)}$	$m_3^{(2)}$	$n_1^{(2)}$	$n_2^{(2)}$	$n_{3}^{(2)}$
-0.8692		-0.6257	0.6195	-0.4468	0.4596	0.5377	0.1777	0.0071	0.3121

Table 9



Fig. 17. Cup drawing simulation for the fictitious material; a) equivalent plastic strain distribution, b) cup height variation along the circumference.

experimental data, correspondingly expanded to retain more or less terms, the BBC2008 plane-stress yield criterion has proved to cope well with the complexity of highly anisotropic plastic behavior. Of course, to become this statement true a proper material characterization had to be done. For that purpose a robust procedure for the identification of the respective model parameters is described in the paper. The constitutive equations were integrated numerically by applying the NICE integration scheme which has proved to be stable, accurate and computationally efficient. Both the constitutive model and the integration scheme have been implemented into ABAQUS/Explicit for further finite element applications.

In order to investigate the capability of the conceived constitutive model in predicting the occurrence of more than four ears in deep drawing of a round cup from highly anisotropic material, the sheet metal forming process was numerically simulated with ABAQUS/Explicit and shell S4R elements used in the finite element discretization. Undoubtedly, great potential of the BBC2008 model is proved by the obtained simulation results. Based on the quite good agreement found between the simulated and experimentally obtained earing profiles it can be also concluded, that the presented modeling approach is physically objective and is able to predict correctly the mechanical response of highly anisotropic sheet metals under complex loading conditions. In addition, the fictitious case is introduced in the paper, which shows that the BBC2008 model is able to predict even twelve ears in cup drawing simulation with the formulation, containing 16 parameters for anisotropy description only. The investigated case proves great flexibility of the model despite its simple formulation.

Finally, since the BBC2008 model does not use linear transformations of the stress tensor and absolute value function in the yield criterion formulation, it is very convenient for implementation in the finite element programs. Further, due to the fact that in this paper numerical integration of the evolution equations in the constitutive model is performed in an improved explicit way, the computational efficiency of the entire presented approach should be in general superior in the simulation of sheet metal forming processes comparing to existing techniques.

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Appendix A

The subsequent analysis will be focused on the convexity of the yield surface described by Eqs. (1) and (3). Due to the fact that $\overline{\sigma} \ge 0$, Y > 0, and w > 1, Eq. (1) can be written in the equivalent form

$$\begin{split} \Psi(\sigma_{11},\sigma_{22},\sigma_{12} \,=\, \sigma_{21},Y) &:= \frac{\left[\overline{\sigma}(\sigma_{11},\sigma_{22},\sigma_{12} \,=\, \sigma_{21})\right]^{2k}}{w-1} - \frac{Y^{2k}}{w-1} \\ &= \sum_{r=1}^{s} \left[w^{r-1}\psi_{1}^{(r)} + w^{s-r}\psi_{2}^{(r)} \right] \,=\, 0 \end{split} \tag{A1}$$

where (see also Eq. (3))

$$\begin{split} \psi_{1}^{(r)}(\sigma_{11},\sigma_{22},\sigma_{12}=\sigma_{21},Y) &= \left[L^{(r)}+M^{(r)}\right]^{2k} + \left[L^{(r)}-M^{(r)}\right]^{2k} \\ &- \frac{Y^{2k}}{2sw^{r-1}(w-1)} \\ \psi_{2}^{(r)}(\sigma_{11},\sigma_{22},\sigma_{12}=\sigma_{21},Y) \\ &= \left[M^{(r)}+N^{(r)}\right]^{2k} + \left[M^{(r)}-N^{(r)}\right]^{2k} - \frac{Y^{2k}}{2sw^{s-r}(w-1)}, \\ r=1,\dots,s. \end{split}$$
(A.2)

Eqs. (1) and (A.1) describe the same yield surface, but regarding the current analysis Eq. (A.1) is more convenient because its leftside term Ψ is expressed as a positive and linear combination of two sets of functions, namely $\psi_1^{(r)}$ and $\psi_2^{(r)}$ (r=1,...,s). In general, the convexity of each of these functions, $\psi_1^{(r)}$ and $\psi_2^{(r)}$ (r=1,...,s), in the space of the stress components σ_{11} , σ_{22} , and $\sigma_{12} = \sigma_{21}$ is a sufficient condition ensuring that Ψ has the same characteristic (Rockafellar, 1970).

One should also notice that $\psi_1^{(r)}$ and $\psi_2^{(r)}$ (r = 1, ..., s) are twice differentiable with respect to variables σ_{11} , σ_{22} , and $\sigma_{12} = \sigma_{21}$. Any function having this property is convex if and only if its Hessian matrix is positive semi-definite (Rockafellar, 1970). The manipulation of the Hessian matrices is difficult if $\psi_1^{(r)}$ and $\psi_2^{(r)}$ (r = 1, ..., s)are formulated in terms of the actual stress components σ_{11} , σ_{22} , and $\sigma_{12} = \sigma_{21}$. This task can be simplified by taking into account the fact that linear transformations preserve the convexity (Rockafellar, 1970). In other words, the positive semi-definite character of the Hessian matrix corresponding to any of the functions $\psi_1^{(r)}$ and $\psi_2^{(r)}$ (r = 1, ..., s) may be analyzed in a different space of variables, say ξ_{1,ξ_2,ξ_3} , related to the actual stress components $\sigma_{11,\sigma_{22},\sigma_{12}} = \sigma_{21}$ by properly chosen linear transformations.

Convexity of functions $\psi_1^{(r)}$ (r = 1, ..., s)

Before proceeding to the computation of the Hessian matrix, it is convenient to reformulate functions $\psi_1^{(r)}$ (r = 1, ..., s) in terms of the following variables (see Eqs. (A.2) and (3)):

$$\begin{aligned} \xi_1 &= \mathfrak{k}_1^{(r)} \sigma_{11} + \mathfrak{k}_2^{(r)} \sigma_{22}, \quad \xi_2 &= m_1^{(r)} \sigma_{11} - m_2^{(r)} \sigma_{22}, \quad \xi_3 \\ &= m_3^{(r)} (\sigma_{12} + \sigma_{21}). \end{aligned} \tag{A.3}$$

By considering the above transformations functions $L^{(r)}$ and $M^{(r)}$ in Eq. (3) become

$$L^{(r)} = \xi_1, \quad M^{(r)} = \sqrt{\xi_2^2 + \xi_3^2}.$$
 (A.4)

The components of the Hessian matrix $[H]_{3\times3}$ associated to any of functions $\psi_1^{(r)}$ (r = 1, ..., s) are easily derivable from Eqs. (A.2) and (A.4), by using the chain rule. The positive semidefinite character of $[H]_{3\times3}$ is ensured if the following inequalities are valid for all values of variables ξ_1 , ξ_2 and ξ_3 (Rockafellar, 1970):

$$\begin{aligned} H_{11} &= \frac{\partial^2 \psi_1^{(r)}}{\partial \xi_1 \partial \xi_1} &= \frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial L^{(r)}} \ge 0\\ \det \left[H_{\alpha\beta} \right]_{\alpha,\beta=1,2} &= \det \left[\frac{\partial^2 \psi_1^{(r)}}{\partial \xi_\alpha \partial \xi_\beta} \right]_{\alpha,\beta=1,2} = \frac{\xi_3^2}{[M^{(r)}]^3} \frac{\partial \psi_1^{(r)}}{\partial M^{(r)}} \frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial L^{(r)}} \\ &+ \frac{\xi_2^2}{\left[M^{(r)} \right]^2} \left\{ \frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial L^{(r)}} \frac{\partial^2 \psi_1^{(r)}}{\partial M^{(r)} \partial M^{(r)}} - \left[\frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial M^{(r)}} \right]^2 \right\} \ge 0\\ \det \left[H_{ij} \right]_{i,j=1,2,3} &= \det \left[\frac{\partial^2 \psi_1^{(r)}}{\partial \xi_i \partial \xi_j} \right]_{i,j=1,2,3} \\ &= \frac{1}{M^{(r)}} \frac{\partial \psi_1^{(r)}}{\partial M^{(r)}} \left\{ \frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial L^{(r)}} \frac{\partial^2 \psi_1^{(r)}}{\partial M^{(r)} \partial M^{(r)}} - \left[\frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial M^{(r)}} \right]^2 \right\} \ge 0. \end{aligned}$$

$$(A.5)$$

The expressions of the partial derivatives involved in Eq. (A.5) can be deduced from Eq. (A.2):

$$\frac{\partial \psi_{1}^{(r)}}{\partial M^{(r)}} = 2k \left\{ \left[L^{(r)} + M^{(r)} \right]^{2k-1} - \left[L^{(r)} - M^{(r)} \right]^{2k-1} \right\}$$

$$\frac{\partial^{2} \psi_{1}^{(r)}}{\partial L^{(r)} \partial L^{(r)}} = \frac{\partial^{2} \psi_{1}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} = 2k(2k-1) \left\{ \left[L^{(r)} + M^{(r)} \right]^{2k-2} + \left[L^{(r)} - M^{(r)} \right]^{2k-2} \right\}$$

$$\frac{\partial^{2} \psi_{1}^{(r)}}{\partial L^{(r)} \partial M^{(r)}} = 2k(2k-1) \left\{ \left[L^{(r)} + M^{(r)} \right]^{2k-2} - \left[L^{(r)} - M^{(r)} \right]^{2k-2} \right\}.$$
(A.6)

By combining the above formulae, one obtains

$$\frac{\partial^{2} \psi_{1}^{(r)}}{\partial L^{(r)} \partial L^{(r)}} \frac{\partial^{2} \psi_{1}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} - \left[\frac{\partial^{2} \psi_{1}^{(r)}}{\partial L^{(r)} \partial M^{(r)}}\right]^{2}$$

= $[4k(2k-1)]^{2} [L^{(r)} + M^{(r)}]^{2k-2} [L^{(r)} - M^{(r)}]^{2k-2} \ge 0.$ (A.7)

Due to the property (see Eq. (A.4))

 $M^{(r)} \ge 0, \tag{A.8}$

the following inequality is also satisfied:

$$L^{(r)} + M^{(r)} \ge L^{(r)} - M^{(r)}.$$
(A.9)

As a consequence,

$$\left[L^{(r)} + M^{(r)}\right]^{2k-1} \ge \left[L^{(r)} - M^{(r)}\right]^{2k-1}$$
(A.10)

if *k* is a strictly positive integer number. The constraint $k \in \mathbb{N} \setminus \{0\}$ is thus sufficient for having (see Eqs. (A.6) and (A.7))

$$\frac{\partial \psi_1^{(r)}}{\partial M^{(r)}} \ge \mathbf{0}, \quad \frac{\partial^2 \psi_1^{(r)}}{\partial L^{(r)} \partial L^{(r)}} \ge \mathbf{0}. \tag{A.11}$$

When analyzing Eqs. (A.5)–(A.11), one may easily notice that $k \in \mathbb{N} \setminus \{0\}$ ensures the positive semi-definite character of the Hessian matrix associated to any of functions $\psi_1^{(r)}$ (r = 1, ..., s) in the (ξ_1, ξ_2, ξ_3) space. Due to linearity of the transformations given by Eq. (A.3), this conclusion remains valid in the space of the actual stress components ($\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}$).

Convexity of functions $\psi_2^{(r)}$ (r = 1, ..., s)

As in the previous analysis, it is convenient to reformulate functions $\psi_2^{(r)}$ (r = 1, ..., s) in terms of the following variables (see Eqs. (A.2) and (3)):

$$\begin{aligned} \xi_1 &= m_1^{(r)} \sigma_{11} - m_2^{(r)} \sigma_{22}, \quad \xi_2 &= n_1^{(r)} \sigma_{11} - n_2^{(r)} \sigma_{22}, \quad \xi_3 \\ &= \sigma_{12} + \sigma_{21} \end{aligned} \tag{A.12}$$

which transforms functions $M^{(r)}$ and $N^{(r)}$ in Eq. (3) into

$$M^{(r)} = \sqrt{\xi_1^2 + \left[m_3^{(r)}\xi_3\right]^2}, \quad N^{(r)} = \sqrt{\xi_2^2 + \left[n_3^{(r)}\xi_3\right]^2}.$$
 (A.13)

The positive semi-definite character of the Hessian matrix $[H]_{3\times 3}$ associated to any of the functions $\psi_2^{(r)}$ (r = 1, ..., s) is ensured if the following inequalities are valid for all values of variables ξ_1 , ξ_2 and ξ_3 :

$$\begin{split} H_{11} &= \frac{\partial^{2} \psi_{2}^{(r)}}{\partial \xi_{1} \partial \xi_{1}} = \frac{[m_{3}^{(r)} \xi_{3}]^{2}}{[M^{(r)}]^{3}} \frac{\partial \psi_{2}^{(r)}}{\partial M^{(r)}} + \frac{\xi_{1}^{2}}{[M^{(r)}]^{2}} \frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} \ge 0\\ \det \left[H_{\alpha\beta} \right]_{\alpha,\beta=1,2} &= \det \left[\frac{\partial^{2} \psi_{2}^{(r)}}{\partial \xi_{\alpha} \partial \xi_{\beta}} \right]_{\alpha,\beta=1,2} \\ &= \frac{\xi_{1}^{2} \xi_{2}^{2}}{\left[M^{(r)} N^{(r)} \right]^{2}} \left\{ \frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} \frac{\partial^{2} \psi_{2}^{(r)}}{\partial N^{(r)} \partial N^{(r)}} - \left[\frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial N^{(r)}} \right]^{2} \right\} \\ &+ \frac{\xi_{3}^{2}}{\left[M^{(r)} N^{(r)} \right]^{3}} \left\{ \left[n_{3}^{(r)} \xi_{1} \right]^{2} M^{(r)} \frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} \frac{\partial \psi_{2}^{(r)}}{\partial N^{(r)}} \right. \\ &+ \left[m_{3}^{(r)} \xi_{2} \right]^{2} N^{(r)} \frac{\partial^{2} \psi_{2}^{(r)}}{\partial N^{(r)} \partial N^{(r)}} \frac{\partial \psi_{2}^{(r)}}{\partial M^{(r)}} \\ &+ \left[m_{3}^{(r)} n_{3}^{(r)} \xi_{3} \right]^{2} \frac{\partial \psi_{2}^{(r)}}{\partial M^{(r)}} \frac{\partial \psi_{2}^{(r)}}{\partial N^{(r)}} \right\} \ge 0 \\ \det \left[H_{ii} \right]_{i,i=1,2,2} &= \det \left[\frac{\partial^{2} \psi_{2}^{(r)}}{\partial T^{(r)}} \right] \end{split}$$

$$= \frac{1}{\left[M^{(r)}N^{(r)}\right]^{2}} \left\{ \left[m_{3}^{(r)}\xi_{2}\right]^{2}M^{(r)}\frac{\partial\psi_{2}^{(r)}}{\partial M^{(r)}} + \left[n_{3}^{(r)}\xi_{1}\right]^{2}N^{(r)}\frac{\partial\psi_{2}^{(r)}}{\partial N^{(r)}} \right\} \times \left\{ \frac{\partial^{2}\psi_{2}^{(r)}}{\partial M^{(r)}\partial M^{(r)}}\frac{\partial^{2}\psi_{2}^{(r)}}{\partial N^{(r)}\partial N^{(r)}} - \left[\frac{\partial^{2}\psi_{2}^{(r)}}{\partial M^{(r)}\partial N^{(r)}}\right]^{2} \right\} + \frac{\left[m_{3}^{(r)}n_{3}^{(r)}\xi_{3}\right]^{2}}{\left[M^{(r)}N^{(r)}\right]^{3}}\frac{\partial\psi_{2}^{(r)}}{\partial M^{(r)}}\left\{ \left[M^{(r)}\right]^{2}\frac{\partial^{2}\psi_{2}^{(r)}}{\partial M^{(r)}\partial M^{(r)}} + 2M^{(r)}N^{(r)}\frac{\partial^{2}\psi_{2}^{(r)}}{\partial M^{(r)}\partial N^{(r)}} + \left[N^{(r)}\right]^{2}\frac{\partial^{2}\psi_{2}^{(r)}}{\partial N^{(r)}\partial N^{(r)}}\right\} \ge 0.$$
(A.14)

The expressions of the partial derivatives involved in Eq. (A.14) can be deduced from Eq. (A.2):

$$\begin{aligned} &\frac{\partial \psi_{2}^{(r)}}{\partial M^{(r)}} = 2k \left\{ \left[M^{(r)} + N^{(r)} \right]^{2k-1} + \left[M^{(r)} - N^{(r)} \right]^{2k-1} \right\} \\ &\frac{\partial \psi_{2}^{(r)}}{\partial N^{(r)}} = 2k \left\{ \left[M^{(r)} + N^{(r)} \right]^{2k-1} - \left[M^{(r)} - N^{(r)} \right]^{2k-1} \right\} \\ &\frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} = \frac{\partial^{2} \psi_{2}^{(r)}}{\partial N^{(r)} \partial N^{(r)}} = 2k(2k-1) \left\{ \left[M^{(r)} + N^{(r)} \right]^{2k-2} + \left[M^{(r)} - N^{(r)} \right]^{2k-2} \right\} \\ &\frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial N^{(r)}} = 2k(2k-1) \left\{ \left[M^{(r)} + N^{(r)} \right]^{2k-2} - \left[M^{(r)} - N^{(r)} \right]^{2k-2} \right\}. \end{aligned}$$

$$(A.15)$$

By combining the above formulae, one obtains

$$\frac{\partial^2 \psi_2^{(r)}}{\partial M^{(r)} \partial M^{(r)}} \frac{\partial^2 \psi_2^{(r)}}{\partial N^{(r)} \partial N^{(r)}} - \left[\frac{\partial^2 \psi_2^{(r)}}{\partial M^{(r)} \partial N^{(r)}}\right]^2$$
$$= \left[4k(2k-1)\right]^2 \left[M^{(r)} + N^{(r)}\right]^{2k-2} \left[M^{(r)} - N^{(r)}\right]^{2k-2} \ge 0$$
(A.16)

and

$$\begin{split} \left[M^{(r)} \right]^{2} & \frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial M^{(r)}} + 2M^{(r)} N^{(r)} \frac{\partial^{2} \psi_{2}^{(r)}}{\partial M^{(r)} \partial N^{(r)}} + \left[N^{(r)} \right]^{2} \frac{\partial^{2} \psi_{2}^{(r)}}{\partial N^{(r)} \partial N^{(r)}} \\ &= 2k(2k-1) \left\{ \left[M^{(r)} + N^{(r)} \right]^{2k} + \left[M^{(r)} - N^{(r)} \right]^{2k} \right\}. \end{split}$$
(A.17)

Due to the property (see Eq. (A.13))

$$M^{(r)} \ge 0, \quad N^{(r)} \ge 0,$$
 (A.18)

the following inequalities are also satisfied:

$$M^{(r)} + N^{(r)} \ge M^{(r)} - N^{(r)}, \quad N^{(r)} + M^{(r)} \ge N^{(r)} - M^{(r)}.$$
 (A.19)

As a consequence,

$$\begin{split} \left[M^{(r)} + N^{(r)} \right]^{2k-1} &\ge \left[M^{(r)} - N^{(r)} \right]^{2k-1}, \quad \left[M^{(r)} + N^{(r)} \right]^{2k-1} \\ &\ge - \left[M^{(r)} - N^{(r)} \right]^{2k-1}, \end{split}$$
(A.20)

if k is a strictly positive integer number. The constraint $k \in \mathbb{N} \setminus \{0\}$ is thus sufficient for having (see Eqs. (A.15)–(A.17))

$$\frac{\partial \psi_2^{(r)}}{\partial M^{(r)}} \ge 0, \quad \frac{\partial \psi_2^{(r)}}{\partial N^{(r)}} \ge 0, \quad \frac{\partial^2 \psi_2^{(r)}}{\partial M^{(r)} \partial M^{(r)}} = \frac{\partial^2 \psi_2^{(r)}}{\partial N^{(r)} \partial N^{(r)}} \ge 0.$$
 (A.21)

One may conclude, on the basis of Eqs. (A.14)–(A.21), that $k \in \mathbb{N} \setminus \{0\}$ is a sufficient condition for ensuring the positive semidefinite character of the Hessian matrix associated to any of the functions $\psi_2^{(r)}$ (r = 1, ..., s) in the (ξ_1, ξ_2, ξ_3) space. Due to linearity of the transformations given by Eq. (A.12), this property will be preserved in the space of the actual stress components ($\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}$).

As according to Eq. (A.1) function Ψ is a linear and positive combination of functions $\psi_1^{(r)}$ and $\psi_2^{(r)}$ (i = 1, ..., s), the condition $k \in \mathbb{N} \setminus \{0\}$ ensures the convexity of the yield surface described by Eqs. (1) and (3). From this point of view there is no constraint acting on the admissible values of the other material parameters included in the expression of the equivalent stress.

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